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A Modified Wide Angle Parabolic Wave Equation

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We demonstrate the implicit finite difference discretization of a higher order parabolic-like partial differential equation approximating the reduced wave equation in the far field and show that the discretization is unconditionally stable We discuss a method of associating an angle of dispersion with parabolic approximations, present an example which can be used to compare methods on the basis of dispersion angle, and make comparisons among well-known methods and the new method.

I. INTRODUCTION

The technique of approximating the reduced wave equation by a Schrödinger-type partial differential equation (a "parabolic approximation") is the main tool used in the area of underwater acoustics for the modeling of wave propagation over long distances in the ocean. This technique has also been utilized in the derivation of one-way wave equations to be used as absorbing boundary conditions [2], in the study of water waves [8], in seismic waves, in quantum mechanics, and in other areas. A significant saving in computer time, memory, and indeed feasibility exists in the numerical solution of the parabolic-like Schrödinger equation over that of the elliptic reduced wave equation.

Parabolic partial differential equations approximating the reduced wave equation are generally derived via a pseudo-partial differential equation containing a square-root operator. The square-root operator is approximated in some fashion and one then arrives at the Schrödinger-type partial differential equation. In the context of underwater acoustics, historically, the "small angle" parabolic wave equation,

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recognized as the standard PE, was first introduced by Tappert and Hardin (see, e.g., [11]) and solved by a "split-step" technique utilizing a fast Fourier transform. A "wide angle" parabolic wave equation, which in a certain sense encompasses the standard PE, was introduced by Claerbout [1] using a rational approximation to the square-root operator. Estes and Fain [3] pursued the solution of the wide angle equation using a fast Fourier transform approach after expanding the denominator of the rational function by an approximating series. Lee and others [5–7] have solved the wide angle equation by an implicit finite difference scheme identified as the "IFD" model. Greene [4] has also pursued the finite difference approach to the wide angle equation.

The names "small angle" and "wide angle" derive from the use of successively better approximations to the square-root operator, namely a two-term Taylor polynomial and a second-order Padé approximation, and relate the detection of energy at successively greater dispersion angles. In this paper we have elected to pursue the three-term Taylor (3TT) approximation. The new approximating partial differential equation is presented, and using implicit finite difference methods the corresponding discrete system is derived. The resulting partial differential equation is a fourth-order parabolic-like equation. The theoretical development is outlined, the numerical stability of the discrete system is proved, the wide angle capability is discussed, and an exact solution is used as a reference in order to examine the accuracy and validity of this new developent and to compare it to existing methods.

II. A Wide Angle Parabolic Approximation

In the context of the usual derivation of a parabolic-like differential equation [5] one obtains a pseudo-partial differential equation of the form

$$u_r + ik_0\left(1 - \left[1 + (n^2 - 1) + (1/k_0^2)\frac{\partial^2}{\partial z^2}\right]^{1/2}\right)u = 0,$$

with associated boundary conditions $u(r, 0) = 0$, and $u_z(r, B) = 0$, for all $r > 0$ (these correspond to pressure release at the ocean surface and a hard bottom), and a source condition $u(r_0, z) = f(z)$. Here $n = n(r, z)$ is the index of refraction and k_0 is a reference wave number. The square-root operator is then approximated Let $L = (n^2 - 1) + (1/k_0^2)(\partial^2/\partial z^2)$; then the part of the pseudo-partial differential equation in the square-root becomes $1 + L$. The classical approximations are

$$\sqrt{1 + L} = 1 + (1/2)\, L + O(L^2)$$

and

$$\sqrt{1 + L} = \frac{(1 + 3/4)\, L)}{(1 + (1/4)\, L)} + O(L^3),$$

they yield the "small angle" equation

$$u_r = \frac{ik_0}{2}\left[(n^2-1) + (1/k_0^2)\frac{\partial^2}{\partial z^2}\right]u, \tag{1}$$

and the "wide angle" equation

$$[1+(1/4)L]\,u_r = \frac{ik_0}{2}Lu, \qquad L = (n^2-1) + (1/k_0^2)\frac{\partial^2}{\partial z^2}, \tag{2}$$

respectively. Equation (2) is a third-order mixed partial differential equation arising essentially as a result of a third order Taylor approximation (via a Padé (1–1) rational function approximation). Of course one can use a third-order Taylor approximation directly in the approximation of the square-root operator. This is the approach we wish to pursue in this development,

$$\sqrt{1+L} = 1 + (\tfrac{1}{2})\,L - (\tfrac{1}{8})\,L^2 + O(L^3).$$

This approximation yields a fourth order partial differential equation

$$u_r = ik_0\left(\frac{1}{2}T\left[1-\frac{1}{4}T\right] + \frac{1}{2k_0^2}\left[1-\frac{1}{2}T\right]\frac{\partial^2}{\partial z^2} - \frac{1\partial^4}{8k_0^4\partial z^4}\right)u - \frac{i}{8k_0}u\frac{\partial^2 T}{\partial z^2},$$
$$T = n^2(r,z) - 1,$$

and in the case of an isotropic medium we obtain

$$u_r = ik_0\left(\frac{1}{2}T\left[1-\frac{1}{4}T\right] + \frac{1}{2k_0^2}\left[1-\frac{1}{2}T\right]\frac{\partial^2}{\partial z^2} - \frac{1\partial^4}{8k_0^4\partial z^4}\right)u. \tag{3}$$

See also [9, 12] for additional discussions of higher order parabolic approximations. We remark that the coefficient of the L^3 term of the error in the Padé (1–1) approximation to $\sqrt{1+L}$ is $\frac{1}{32}$ and the corresponding coefficient in the 3TT expansion is $\frac{1}{16}$.

III. Finite Difference Discretization

We shall apply a Crank–Nicolson type method to discretize (3). We shall use a rectangular wave guide in (r, z) space, $0 \leqslant z \leqslant B$, and $0 < r_0 \leqslant r \leqslant r \leqslant R$, $z = 0$ is the ocean surface and $z = B$ the bottom. We choose the standard grid on the wave guide, $h = \Delta z$, $M\,\Delta z = B$, M an integer, $k = \Delta r$, and for $z_m = mh$, $r_n = r_0 + nk$, m, n integers, $u(r_n, z_m) = u_m^n$; thus z_0 is the surface and z_M is the bottom. We shall use the letter "n" in two different ways, as a counter on the range variable r and to designate the index of refraction; the context will make it clear which is intended in each case. A standard way in which the Crank-Nicolson approximation is derived for traditional parabolic partial differential equations is to take the average of the classical explicit (forward) difference approximation based at the point (r_n, z_m) and the (backwards) implicit approximation based at (r_{n+1}, z_m). We employ this approach here. The discretization is presented for a partial differential equation of the form

$$u_r = \alpha(r, z)\, u + \beta(r, z) \frac{\partial^2 u}{\partial z^2} + \gamma(r, z) \frac{\partial^4 u}{\partial z^4}, \tag{4}$$

where the coefficients are assumed to be complex valued. We use standard three- and five-point centered difference formulas to approximate second and fourth derivatives with respect to z. The resulting linear system has its general equation involving five of the unknowns

$$\begin{aligned} &-e_m^{n+1} u_{m-2}^{n+1} - a_m^{n+1} u_{m-1}^{n+1} + (2 - d_m^{n+1})\, u_m^{n+1} - a_m^{n+1} u_{m+1}^{n+1} - e_m^{n+1} u_{m+2}^{n+1} \\ &\quad = e_m^n u_{m-2}^n + a_m^n u_{n-1}^n + (2 + d_m^n)\, u_m^n + a_m^n u_{m+1}^n + e_m^n u_{m+2}^n. \end{aligned} \tag{5}$$

$m = 1, 2, \ldots, M$; $n = 1, 2, \ldots,$ where

$$e_m^n = \frac{k_0 \gamma_m^n}{h^4}, \qquad a_m^n = \frac{k_0}{h^2} \left[\beta_m^n - \frac{4\gamma_m^n}{h^2} \right], \qquad d_m^n = k_0 \left[\alpha_m^n - \frac{2\beta_m^n}{h^2} + \frac{6\gamma_m^n}{h^4} \right].$$

IV. Boundary Conditions and Matrix Formulation

It is important to translate the boundary conditions related to the second-order derivatives in z in the original problem into corresponding ones for the new equation which is a fourth-order equation in z. The original problem is to solve the reduced wave equation (Helmholtz equation in cylindrical coordinates with the θ variable suppressed) for an outgoing wave in the far fields with a point source driver, i.e.,

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k_0^2 n^2 p = -\delta(z - z_s) \frac{\delta(r)}{2\pi r} \tag{6}$$

with pressure release surface and hard bottom. Here p represents the acoustic pressure of a point source located at the point $r = 0$, $z = z_s$ in the ocean. Simplified equations are obtained from (6) by substituting

$$p(r, z) = u(r, z)\, H_0^{(1)}(k_0 r),$$

where $H_0^{(1)}(r)$ is the Hankel function of the first kind of order zero, and making various approximations to obtain an equation for the envelope u. The far-field approximation (asymptotically approximating the Hankel function) leads to the elliptic envelope equation

$$\frac{\partial^2 u}{\partial r^2} + 2ik_0 \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + k_0^2(n^2 - 1)\, u = 0. \tag{7}$$

The pseudo-partial differential equation is then obtained as an approximation to (7). Now the condition $u(r, 0) = 0$ for all r implies that all higher order derivatives of $u(r, z)$ with respect to r are also zero at $z = 0$, it follows on evaluating (7) at $(r, 0)$ that $u_{zz}(r, 0) = 0$. In addition if one differentiates (7) with respect to z and uses the condition $u_z(r. B) = 0$, one obtains $u_{zzz}(r, B) = 0$ for all r. Thus in addition to the basic boundary conditions

$$u(r, 0) = 0, \qquad u_z(r, B) = 0.$$

We have inherited boundary conditions

$$u_{zz}(r, 0) = 0, \qquad u_{zzz}(r, B) = 0.$$

The discretization of these conditions proceeds as follows: $u(r, 0) = 0$ translates to $u_0^n = 0$, $n \times 1, 2,...$ and introducing the ficticious point u_{-1}^n and a central difference to approximate $u_{zz}(r, 0) = 0$, one obtains $u_{-1}^n = -u_1^n$, $n = 1, 2,....$ To discretize the bottom conditions one introduces the points u_{M+1}^n and u_{M+2}^n. Using a central difference on $u_z(r, B) = 0$ yields $u_{M+1}^n = u_{M-1}^n$. It then follows on approximating $u_{zzz}(r, B) = 0$ that $u_{M+2}^n = u_{M-2}^n$, $n = 1, 2,....$

The active unknowns are u_m^n, $m = 1, 2,...., M$. Thus the system of Eqs. (5) needs to be modified in its first row and its last two rows to take the boundary conditions into consideration. The general system has the form

$$(I - \tfrac{1}{2}A^{(n+1)})\, u^{n+1} = (I + \tfrac{1}{2}A^{(n)})\, u^n \tag{8}$$

where

$$A = \begin{bmatrix} d'_1 & a_1 & e_1 & & & & & \\ a_2 & d_2 & a_2 & e_2 & & & & \\ e_3 & a_3 & d_3 & a_3 & e_3 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & e_{M-2} & a_{M-2} & d_{M-2} & a_{M-2} & e_{M-2} \\ & & & & e_{M-1} & a_{M-1} & d'_{M-1} & a_{M-1} \\ & & & & & e'_M & a'_M & d_M \end{bmatrix}$$

and $d'_1 = d_1 - e_1$, $d'_{M-1} = d_{M-1} + e_{M-1}$, $e'_M = 2e_M$, $a'_M = 2a_M$. The superscripts n and $n+1$ on A indicate the range points at which the coefficients are to be evaluated.

In the application to (3) the coefficients in the matrix system are

$$e^n_m \equiv e = -\frac{ik_0}{8k_0^3h^4}, \qquad a^n_m = \frac{ik_0}{2k_0h^2}\left[1 - \frac{1}{2}T^n_m + \frac{1}{h^2k_0^2}\right],$$
$$d^n_m = ik_0\left[\frac{T^n_m}{2}\left(k_0 + \frac{1}{k_0h^2} - \frac{k_0}{4}T^n_m\right) - \frac{1}{k_0h^2}\left(1 + \frac{3}{4k_0^2h^2}\right)\right]. \tag{9}$$

Since each of these is a pure imaginary number and the main difference between the left and right sides of (8) is a minus sign, it follows, at least in the case of range independence, that in (8) the coefficient of u^{n+1} is the complex conjugate of that of u^n.

V. Stability

As a simple illustration, we choose to discuss the stability of the system in the homogeneous medium case, i.e., the coefficient functions in (4) are assumed constant. We shall use the matrix system (8) which encompasses both the system (5) and the boundary conditions. We need to show that u^n is bounded for all $n = 1, 2,...$ The coefficient matrix A is not symmetric (we are using the term "symmetric" here intentionally) as a result of the boundary conditions but we can transform the system (8) to an equivalent system having symmetric coefficient matrices. Let P be the $M \times M$ diagonal matrix having diagonal entries $[1,..., 1, 1/\sqrt{2}]$, then $B = PAP^{-1}$ is a symmetric five diagonal matrix. It follows that the system (8) can be written in the form

$$P^{-1}(I - \tfrac{1}{2}B)\, Pu^{n+1} = P^{-1}(I + \tfrac{1}{2}B)\, Pu^n$$

and thus we have a symmetric problem

$$(I - \tfrac{1}{2}B)\, v^{n+1} = (I + \tfrac{1}{2}B)\, v^n \tag{10}$$

where $u^{n+1} = P^{-1}v^{n+1}$. Clearly u^n is bounded for all n if and only if v^n is. Now (10) implies that

$$v^n = [(I - \tfrac{1}{2}B)^{-1}(I + \tfrac{1}{2}B)]^n\, v^0$$

and thus to verify stability it suffices to show that $\|(I - \frac{1}{2}B)^{-1}(I + \frac{1}{2}B)\| \leqslant 1$.

We shall pursue this in the specific case of the application, namely for the coefficients in (9). We continue to assume a homogeneous medium, i.e., $n(r, z)$ constant. Recall, we have observed that in this case the coefficient matrices in (8) are complex conjugates of each other. The similarity transform P does not disturb this property and thus the coefficient matrices in (10) are complex conjugates of each other. In fact the matrix B is pure imaginary. Let (10) be written in the form

$$(I - iW)\, v^{n+1} = (I + iW)\, v^n,$$

then W is a real symmetric matrix. Now the matrix $\mathscr{A} = I + iW$ is *normal* since

$$\mathscr{A}^* \mathscr{A} = (I + iW)^*(I + iW) = (I - iW)(I + iW) = I + W^2 = \mathscr{A}\mathscr{A}^*.$$

But for a nonsingular normal matrix $\mathscr{A}$, each of the eigenvalues of $\mathscr{A}^{*-1}\mathscr{A}$ has magnitude unity and $\mathscr{A}^{*-1}\mathscr{A}$ has a complete orthonormal set of eigenvectors, it thus follows [10], on verifying that $\mathscr{A}$ is nonsingular, that

$$\|\mathscr{A}^{*-1}\mathscr{A}\| = \|(I - \tfrac{1}{2}B)^{-1}(I + \tfrac{1}{2}B)\| \leqslant 1.$$

It is obvious that $\mathscr{A}$ and $\mathscr{A}^*$ are nonsingular since if γ is an n-vector of magnitude 1 such that $\mathscr{A}\gamma = 0$ then

$$0 = \gamma^* \mathscr{A} \gamma = 1 + i\gamma^* W \gamma,$$

which is impossible since $\gamma^* W \gamma$ is real (recall W is real symmetric). Thus we have verified that *the Crank–Nicolson type scheme with boundary conditions for* (3) *is unconditionally stable in the case of a homogeneous medium.*

VI. Propagation Angle and the Equation

In this section we wish to consider a criterion for determining maximum propagation angle associated with a given parabolic approximating partial differential equation [4]. (In the next section we will discuss how one can relate angles of propagation to a specific solution of (6).) The criterion employs the dispersion relation of the reduced wave equation

$$k_r^2 + k_z^2 = 1,$$

where k_r is the cosine of the angle of propagation, θ, measured from the horizontal (thus the full angle of propagation is twice θ), and k_z is the cosine of its complement, $k_z = \sin(\theta)$. The question of measuring the maximum angle of propagation then is answered by applying the function being used to approximate the square root in the dispersion relation, i.e., if $g(x)$ is being used to approximate $\sqrt{1 + x}$, $x = -k_z^2 = -\sin^2 \theta$ then

$$k_r = \pm \sqrt{1 - k_z^2} \approx g(-k_z^2),$$

and we think of $g(-k_z^2)$ as an approximation to k_r. We then define the approximate dispersion relation

$$[g(-k_z^2)]^2 + k_z^2 = 1 + \tau,$$

where τ is the tolerance. One then takes for the maximum propagation angle, the largest angle θ which satisfies the approximate dipersion relation while maintaining τ under a specified tolerance. In the ensuing development we have maintained a tolerance of 0.00025. (Of course, using a different tolerance would yield a different maximum angle.) The original angle of propagation is given by $\theta = \arctan(k_z/k_r)$ so the approximate angle of propagation is then given by $\arctan(k_z/g(-k_z^2))$.

This process yields for the best known approximating functions, namely, that used in PE, $1 + (\frac{1}{2})x$, a maximum angle of 10°, and mthat used in WA, $(1 + (\frac{3}{4})x)/(1 + (\frac{1}{4})x)$, an angle of 23°. The three-term Taylor approximation yields 20°

VII. Comparison of the Methods

We choose to compare the two wide-angle parabolic equations by considering a simple problem in a homogeneous medium. The essence of the parabolic method is to approximate solutions of the Helmholtz equation (6), so we shall compare how well the various parabolic equations approximate in the far field the sum of the propagating modes solution of the Helmholtz equation. Since with each such mode one can associate a maximum angle of propagation, it is thus possible to compare the methods on the basis of size of the maximum angle of propagation which a method can detect.

We shall consider (6) with boundary conditions $p(r, 0) = 0$ and $p_z(r, B) = 0$, along with the radiation condition at infinity. Following Ahluwalia and Keller (see, e.g., [0]) the normal mode representation of the solution of (6) is given by

$$p(r, z) = \frac{i}{2B} \sum_{j=0}^{\infty} \psi_j(z_0)\, \psi_j(z)\, H_0^{(1)}(k_0 a_j r),$$

where $H_0^{(1)}$ is the Hankel function, $\psi_j(z) = \sin(k_0 \sqrt{1 - a_j^2}\, z)$, $a_j = \{1 - [(j + \frac{1}{2})(\pi/k_0 B)]^2\}^{1/2}$, B is the depth of the flat bottom ocean, $k_0 = 2\pi f/c_0$, f is the frequency, c_0 is the reference sound speed. Here we have taken the index of refraction $n \equiv 1$. We remark that p is the Green's function for the boundary value problem.

Now as j increases the argument of the Hankel function eventually becomes imaginary and then the terms in the series decay exponentially. This is the condition which determines the number, $N_0 + 1$, of propagating modes, in particular N_0 is the largest integer j for which

$$1 - \left[\left(j + \frac{1}{2}\right)\left(\frac{\pi}{k_0 B}\right)\right]^2 > 0.$$

A given propagating mode corresponds to a system of rays propagating with angle $\pm\theta_j$ about the horizontal, where

$$\theta_j = \frac{\pi}{2} - \tan^{-1}\left(k_0 a_j \Big/ \left(j + \frac{1}{2}\right) \frac{\pi}{B}\right).$$

Thus one can associate a *maximum angle of propagation*, θ_l, with any sum of modes in which case propagation takes place with angles θ, $|\theta| \leqslant \theta_l$.

The test problem we use for purposes of making comparisons is constructed as follows. For a given integer N we take as a benchmark solution

$$p_N(r, z) = \frac{i}{2B} \sum_{j=0}^{N} \psi_j(z_0)\, \psi_j(z)\, H_0^{(1)}(k_0 a_j r),$$

and use $f(z) \equiv p_N(r_0, z)/H_0^{(1)}(k_0 r_0)$ as a source condition for the parabolic solvers. Thus we can speak of $N+1$ modes being propagated, $N \leqslant N_0$, with the maximum angle of propagation being $\theta_l = \pi/2 - \tan^{-1}(k_0 a_{N_0}/(N_0 + \frac{1}{2})\, \pi/B)$ (see Fig. 1).

We have selected parameters which generally correspond to an underwater acoustics application, namely $c_0 = 1500\,\mathrm{m/s}$, $f = 50$ Hz (and thus $k_0 = 0.2094$), $B = 400$ m, $z_s = 300$ m, $r_0 = 5000$ m, and have chosen to monitor the signal at a receiver at depth 100 m. This set of parameters yields a total of 27 propagating modes, $j = 0, 1, ..., 26$. The sum of the first eleven modes corresponds to a maximum angle of propagation of 23.19°, and the sum of the first 13 modes corresponds to an angle of 27.95°. It is these two cases that we have chosen to examine. We have maintained the step sizes in all of the computations at $\Delta z = 1$, and $\Delta r = 1$. If we let $u(r, z)$ denote the computed solution then the plotted curves are of the propagation loss,

$$\mathrm{PL}(r, z) = 10 \log_{10} \left[\frac{1}{|u(r, z)\, H_0^{(1)}(k_0 r)|^2} \right],$$

reflected about the r-axis and relabeled.

In Fig. 1 a comparison is made between (1), (3), and the benchmark solution consisting of the sum of eleven modes, as was anticipated in the forgoing discussion (10° maximum angle and a second-order approximation, as opposed to a 20° maximum angle and a third-order approximation); (1) is not comparable in accuracy to (3).

In Fig. 2, we have again used the sum of the first eleven modes to compare (2) and (3) against the benchmark. Results are quite good, in general, with (2) having a slight edge.

Finally, in Fig. 3 we have chosen to compare (2) and (3) on the 27.95° solution primarily to demonstrate how well the tolerance, 0.00025, predicts the maximum angle of propagation associated with an equation. We observe that even though both methods are tracking the exact solution quite well, an error of magnitude greater than 1db is evident at a number of places and a pronounced phase shift is apparent.

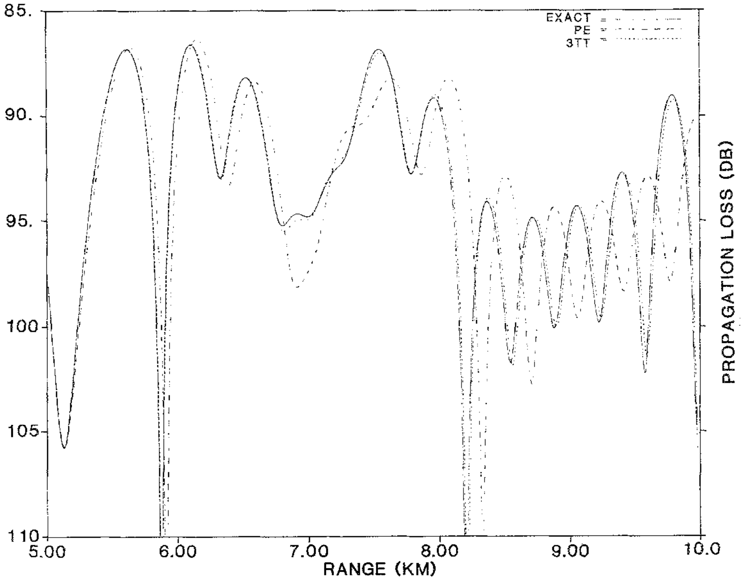

FIG. 1. Comparison at 23.2°.

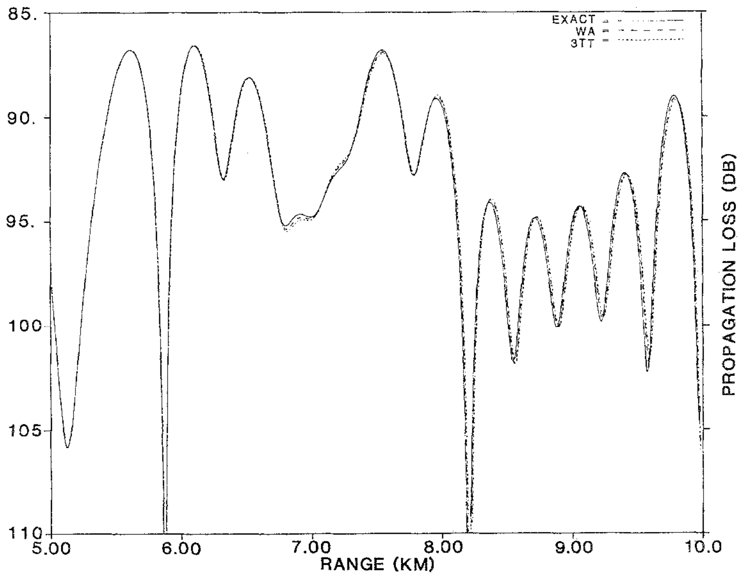

FIG. 2. Comparison at 23.2°.

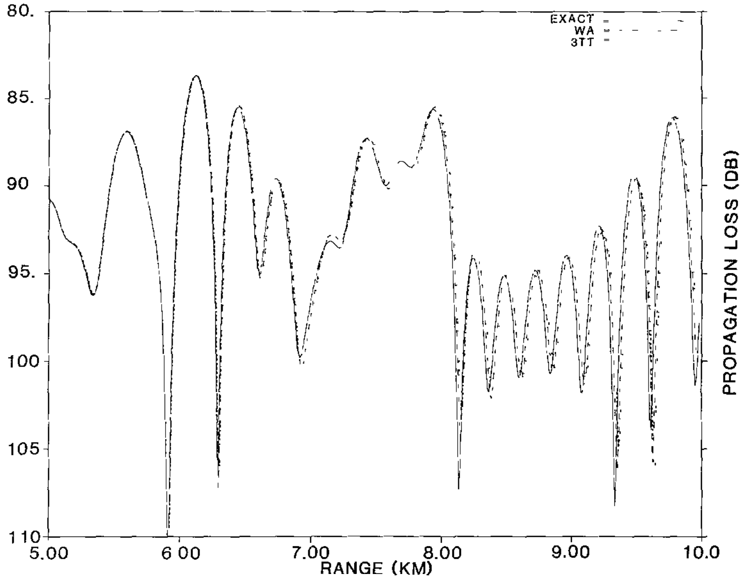

FIG. 3. Comparison at 27 9°.

VIII. CONCLUSIONS

The main contribution of this paper is the demonstration of the discretization of a higher order parabolic-like partial differential equation which may be useful for wide-angle underwater acoustic wave propagation and proof that the numerical scheme is unconditionally stable. The discrete equation results in a five-diagonal linear system to which we apply an easily implemented five-diagonal solver which is very comparable in running time to tri-diagonal solvers. The numerical results are similar to those obtained from the best known second-order parabolic approximation.

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